On Twist Quantizations of $D = 4$ Lorentz and Poincaré Algebras

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Abstract

We use the decomposition of $o(3,1) = sl(2;\mathbb{C})_1 \oplus sl(2;\mathbb{C})_2$ in order to describe nonstandard quantum deformation of $o(3,1)$ linked with Jordanian deformation of $sl(2;\mathbb{C})$. Using twist quantization technique we obtain the deformed coproducts and antipodes which can be expressed in terms of real physical Lorentz generators. We describe the extension of the considered deformation of $D = 4$ Lorentz algebra to the twist deformation of $D = 4$ Poincaré algebra with dimensionless deformation parameter.
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1 Introduction

The quantum deformations of relativistic symmetries are described by Hopf-algebraic deformation of Lorentz and Poincaré algebras. Such quantum deformations are classified by Lorentz and Poincaré Poisson structures. These Poisson structures given by classical r-matrices were classified already some time ago by S. Zakrzewski (see [1] for the Lorentz classical r-matrices and [2] for the Poincaré classical matrices). In [1] there are provided four classical $o(3,1)$ r-matrices and in [2] one finds 21 cases describing different deformation of Poincaré symmetries, with various numbers of free parameters.

In this paper we would like to describe the explicit Hopf algebra form of the nonstandard deformations of $D = 4$ Lorentz algebra and extend it to $D = 4$ Poincaré algebra. We shall describe firstly the twist deformation of complexified $D = 4$ Lorentz algebra $o(4;\mathbb{C})$, and further consider the reality structure ($*-$Hopf algebra) describing quantum deformation of the standard Lorentz algebra $o(3,1)$. Let us observe that the complex Lie

algebra $o(4;\mathbb{C})$ decomposes into a direct sum of two copies of $o(3;\mathbb{C}) \approx sl(2;\mathbb{C})$ algebras. We shall employ the $sl(2;\mathbb{C})_1 \oplus sl(2;\mathbb{C})_2$ basis $(H_1,\, E_1,\, F_1) \oplus (H_2,\, E_2,\, F_2)$ where

$$[H_k,\, E_k] = E_k \qquad [H_k,\, F_k] = -F_k \qquad [E_k,\, F_k] = 2H_k\,, \qquad k = 1,2 \tag{1.1}$$

and obviously

$$[X_1,\, X_2] = 0, \qquad X \in (H,\, E,\, F) \tag{1.2}$$

The real $o(1,3)$ Lorentz algebra is obtained from the reality condition $X_1^\star = -X_2$ or equivalently $X_2^\star = -X_1$. The real generators ($x^\star = -x$, for $x \in (x, x')$), where $X_{1,2} = x \pm ix'$ [1] we describe explicitly as follows

$$\begin{aligned} h &= \frac{1}{2}(H_1 + H_2), \qquad e = \frac{1}{2}(E_1 + E_2), \qquad f = \frac{1}{2}(F_1 + F_2) \\ h' &= \frac{-i}{2}(H_1 - H_2), \qquad e' = \frac{-i}{2}(E_1 - E_2), \qquad f' = \frac{-i}{2}(F_1 - F_2) \end{aligned} \tag{1.3}$$

Notice that these new generators satisfy the commutation relation of the Lorentz algebra $o(1,3)$ written in Cartan-Chevaley basis:

$$\begin{gathered} [h, e] = e\,, \qquad [h, f] = -f\,, \qquad [e, f] = 2h \\ [h, e'] = [h', e] = e'\,, \qquad [h', f] = [h, f'] = -f'\,, \qquad [e, f'] = [e', f] = 2h'\,, \\ [h', e'] = -e\,, \qquad [h', f'] = f\,, \qquad [e', f'] = -2h \end{gathered} \tag{1.4}$$

with vanishing all remaining relations.

We recall that the quantization of classical Lie-algebra g is obtained by introducing the twist function $\mathcal{F} \in U(g) \otimes U(g)$ which modifies the coproduct Δ and antipode S as follows [3]:

$$\Delta \longrightarrow \Delta_{\mathcal{F}} = \mathcal{F}\, \Delta^{(0)}\, \mathcal{F}^{-1}\,, \qquad S \longrightarrow S_{\mathcal{F}} = u\, S\, u^{-1}\,, \tag{1.5}$$

where

$$\begin{gathered} \Delta^{(0)}(x) = x \otimes 1 + 1 \otimes x, \qquad \forall x \in g \\ \mathcal{F} = \sum_i f_i^{(1)} \otimes f_i^{(2)}\,, \qquad u = \sum_i f_i^{(1)}\, S\,(f_i^{(2)})\,. \end{gathered} \tag{1.6}$$

It appears that in classical enveloping Lie algebra $U(g)$, considered as a Hopf algebra $H^{(0)} = (U(g), m, \Delta^{(0)}, S, \epsilon)$ only coalgebra sector (coproduct and coinverse) is modified.

The standard Drinfeld-Jimbo quantum deformation of $D = 4$ Lorentz algebra (see [4, 5]) satisfies modified YB equation and can not be extended to Poincare algebra. In this paper we shall consider the nonstandard quantum deformations satisfying CYBE which can be extended to the whole Poincaré algebra and provide new deformation of relativistic symmetries. We shall consider here more in detail the two-parameter nonstandard deformation generated by the classical $sl(2;\mathbb{C})$ r-matrix $r(\alpha,\beta)$

$$\begin{aligned} r(\alpha,\beta) &= \alpha(h \wedge e - h' \wedge e') + \beta\, e \wedge e' \\ &= \alpha(H_1 \wedge E_1 + H_2 \wedge E_2) + i\beta\, E_1 \wedge E_2\,, \end{aligned} \tag{1.7}$$

[1] We see that besides of the $\star$-operation one can also introduce a standard complex conjugation operation by $\overline{X_1} = X_2$ and $\overline{X_2} = X_1$.

Further we shall assume that the real r-matrix (1.7) is anti-Hermitean, i.e. $H_1^* = -H_2$, $E_1^* = -E_2$ and the parameters α and β should be purely imaginary.

We see that for $\beta = 0$ the r-matrix (1.7) describes the real part of complex Jordanian r-matrix for $sl(2,\mathbb{C}) \approx o(3;\mathbb{C})$, which is quantized by the following Ogievetsky twist ($k = 1, 2$) (see [6])

$$\mathcal{F}_{J,k} = \exp\left(H_k \otimes \Sigma_k\right) \tag{1.8}$$

where $\Sigma_k = ln(1 + \alpha E_k)$. Because the generators $(H_1,\, E_1)$ and $(H_2,\, E_2)$ do commute the twist function corresponding to (1.7) is given by the following formula:

$$\mathcal{F}(\alpha, \beta) = \mathcal{F}_R \mathcal{F}_{J,1} \mathcal{F}_{J,2} \tag{1.9}$$

where

$$\mathcal{F}_R = \exp\left(i\,\beta \Sigma_1 \wedge \Sigma_2\right) \tag{1.10}$$

We would like to mention here that the form of the twist function given above by formula (1.8) was conjecture with not antisymmetrized exponential form (1.10) by Kulish and Mudrov in [7].

In order to deform the real Lorentz algebra we demand that our twist quantization is compatible with $\star$-algebra structure. More explicitly, this means that the twist $\mathcal{F}(\alpha, \beta)$ must be $\star$-unitary, i.e. $\mathcal{F}(\alpha, \beta)^\star = \mathcal{F}(\alpha, \beta)^{-1}$. As we have mentioned above the last equation can be statisfied if and only if both parameters α, β are purely imaginary $(\Sigma_1^\star = \Sigma_2)^2$, i.e. one has

$$(\mathcal{F}_{J,k})^\star = \mathcal{F}_{J,k+1}^{-1}, \qquad (\mathcal{F}_R)^\star = \mathcal{F}_R^{-1}\,. \tag{1.11}$$

In Sect. 2 we shall describe the deformed coproducts in the classical $sl(2;\mathbb{C})_1 \oplus sl(2;\mathbb{C})_2$ basis $(H_1,\, E_1,\, F_1) \oplus (H_2,\, E_2,\, F_2)$ which can be expressed easily in terms of physical (Hermitean) $o(3,1)$ Lorentz algebra basis $(M_i,\, N_i)$, satisfying the algebra

$$[M_i, M_j] = i\epsilon_{ijk} M_k, \qquad [M_i, N_j] = i\epsilon_{ijk} N_k, \qquad [N_i, N_j] = -i\epsilon_{ijk} M_k, \tag{1.12}$$

where using (1.3) one can find that

$$\begin{aligned} h = i\,N_3, \qquad h' &= -i\,M_3 \\ e = i\,(N_1 + M_2), \qquad e' &= i\,(N_2 - M_1) \\ f = i\,(N_1 - M_2), \qquad f' &= -i\,(N_2 + M_1) \end{aligned} \tag{1.13}$$

In Sect. 3 we shall add to the Lorentz algebra (1.12) four momentum generators and describe the quantum deformation of Poincaré algebra, generated by classical r-matrix (1.7) (or equivalently (1.9)). Finally in Sect. 4 we shall present an outlook and mention possible applications.

[2] However $\overline{\Sigma_1} \neq \Sigma_2$.

2 Two-parameter nonstandard deformation of $o(3,1)$ in classical basis

If we use (1.1-1.2), (1.5) and (1.9) we obtain the following formulae for the coproducts of $sl(2;\mathbb{C}) \approx o(3;\mathbb{C})$ generators $(H_k,\ E_k,\ F_k)$, $k=1,2$

$$
\Delta_{\alpha,\beta}(E_k) = \mathcal{F}(\alpha,\beta)\Delta^{(0)}(E_k)\mathcal{F}^{-1}(\alpha,\beta) = \Delta_\alpha(E_k) = E_k \otimes e^{\Sigma_k} + 1 \otimes E_k \tag{2.14}
$$

$$
\begin{aligned}
\Delta_{\alpha,\beta}(H_k) &= H_k \otimes e^{-\Sigma_k} + 1 \otimes H_k - (-1)^k i\beta \Lambda_k e^{\Sigma_k} \otimes \Sigma_{k+1} e^{-\Sigma_k} \\
&\quad + (-1)^k i\beta \Sigma_{k+1} \otimes \Lambda_k e^{\Sigma_k}
\end{aligned} \tag{2.15}
$$

$$
\begin{aligned}
\Delta_{\alpha,\beta}(F_k) &= F_k \otimes e^{-\Sigma_k} + 1 \otimes F_k + 2\alpha H_k \otimes H_k e^{-\Sigma_k} + \alpha H_k(H_k-1) \otimes \Lambda_k + \\
&\quad 2(-)^k i\alpha\beta H_k \Sigma_{k+1} \otimes \Lambda_k - 2(-)^k i\alpha\beta H_k \otimes \Sigma_{k+1} e^{-\Sigma_k} \\
&\quad - 2(-)^k i\alpha\beta H_k \Lambda_k e^{\Sigma_k} \otimes \Sigma_{k+1} e^{-2\Sigma_k} - 2(-)^k i\alpha\beta \Lambda_k e^{\Sigma_k} \otimes H_k \Sigma_{k+1} e^{-\Sigma_k} \\
&\quad + 2(-)^k i\alpha\beta \Sigma_{k+1} \otimes H_k e^{-\Sigma_k} + (-)^k i\alpha\beta \Lambda_k \otimes \Sigma_{k+1} e^{-\Sigma_k} - \\
&\quad (-)^k i\alpha\beta \Sigma_{k+1} \otimes \Lambda_k + (-)^k i\alpha\beta (e^{-2\Sigma_k} - 1) \otimes \Sigma_{k+1}\Lambda_k \\
&\quad - \alpha\beta^2 \Lambda_k \otimes \Sigma_{k+1}^2 e^{-\Sigma_k} - \alpha\beta^2 \Sigma_{k+1}^2 \otimes \Lambda_k - \alpha\beta^2 \Lambda_k^2 e^{2\Sigma_k} \otimes \Sigma_{k+1}^2 \Lambda_k
\end{aligned} \tag{2.16}
$$

Here Σ_{k+1} is understood with index mod 2, i.e. Σ_{k+1} denotes Σ_1 for $k=2$ and $\Lambda_k = e^{-2\Sigma_k} - e^{-\Sigma_k}$.

Using the relations (1.5-1.6) one obtains the following formulae for the antipodes

$$
S_{\alpha,\beta}(E_k) = S_\alpha(E_k) = -E_k e^{-\Sigma_k}, \qquad S_{\alpha,\beta}(H_k) = S_\alpha(H_k) = -H_k e^{\Sigma_k} \tag{2.17}
$$

$$
S_{\alpha,\beta}(F_k) = -F_k e^{\Sigma_k} + \alpha H_k^2 e^{\Sigma_k}(e^{\Sigma_k}+1) - \alpha^2 H_k E_k e^{\Sigma_k} + 2\alpha^3\beta^2 \Sigma_{k+1}^2 E_k^2 \tag{2.18}
$$

In order to apply our results to the physical basis of the Lorentz algebra one has to calculate the coproducts for the generators (h, h', e, e', f, f') by using (1.3), (2.14)-(16) and further the formulae (1.13).

3 Extension of the deformation to Poincaré algebra

The $D=4$ Lorentz algebra (1.12) can be extended to $D=4$ Poincaré algebra by adding the mutually commuting four-momentum operators $(P_0,\ P_1,\ P_2,\ P_3)$ satisfying the relations $(j,k,l=1,2,3)$

$$
\begin{aligned}
[M_j,\ P_l] &= i\,\epsilon_{jlk}\,P_k\ , \qquad [M_j,\ P_0] = 0\ , \\
[N_j,\ P_k] &= -i\,\delta_{jk}\,P_0\ , \qquad [N_j,\ P_0] = -i\,P_j\ .
\end{aligned} \tag{3.19}
$$

Because the classical $r-$matrix (1.7) for $D = 4$ Lorentz algebra satisfies CYBE it provides also the deformation of the $D = 4$ Poincaré algebra. The twisted coproducts of the four-momenta in the "spinorial" basis $P_{\pm} = P_0 \pm P_3$, $P'_{\pm} = P_1 \pm iP_2$ are given by the formulae:

$$\Delta_{\alpha,\beta}(P_+) = \mathcal{F}(\alpha,\beta)\Delta^{(0)}(P_+)\mathcal{F}^{-1}(\alpha,\beta) = \Delta_\alpha(P_+) = P_+ \otimes e^{\frac{1}{2}(\Sigma_1+\Sigma_2)} + 1 \otimes P_+ \tag{3.20}$$

$$\begin{aligned}\Delta_{\alpha,\beta}(P'_-) &= P'_- \otimes e^{\frac{1}{2}(\Sigma_2-\Sigma_1)} + 1 \otimes P'_- + \alpha H_1 \otimes P_+ e^{-\Sigma_1} + \\ &\quad i\alpha\beta P_+ e^{-\Sigma_1} \otimes \Sigma_2 e^{\frac{1}{2}(\Sigma_2-\Sigma_1)} + i\alpha\beta\Lambda_1 e^{\Sigma_1} \otimes \Sigma_2 P_+ e^{-\Sigma_1} - i\alpha\beta\Sigma_2 \otimes P_+ e^{-\Sigma_1}\end{aligned} \tag{3.21}$$

$$\begin{aligned}\Delta_{\alpha,\beta}(P'_+) &= P'_+ \otimes e^{\frac{1}{2}(\Sigma_1-\Sigma_2)} + 1 \otimes P'_+ + \alpha H_2 \otimes P_+ e^{-\Sigma_2} - \\ &\quad i\alpha\beta P_+ e^{-\Sigma_2} \otimes \Sigma_1 e^{\frac{1}{2}(\Sigma_1-\Sigma_2)} - i\alpha\beta\Lambda_2 e^{\Sigma_2} \otimes \Sigma_1 P_+ e^{-\Sigma_2} + i\alpha\beta\Sigma_1 \otimes P_+ e^{-\Sigma_2}\end{aligned} \tag{3.22}$$

$$\begin{aligned}\Delta_{\alpha,\beta}(P_-) &= P_- \otimes e^{-\frac{1}{2}(\Sigma_1+\Sigma_2)} + 1 \otimes P_- + \alpha P'_+ e^{-\Sigma_1} \otimes \Sigma_2 e^{-\frac{1}{2}(\Sigma_1+\Sigma_2)} + \\ &\quad + \alpha H_1 \otimes P'_+ e^{-\Sigma_1} + \alpha H_2 \otimes P'_- e^{-\Sigma_2} + \alpha^2 H_1 H_2 \otimes P_+ e^{-(\Sigma_1+\Sigma_2)} + \\ &\quad - i\alpha\beta P'_- e^{-\Sigma_2} \otimes \Sigma_1 e^{-\frac{1}{2}(\Sigma_1+\Sigma_2)} + i\alpha\beta\Sigma_1 \otimes P'_- e^{-\Sigma_2} + i\alpha\beta\Lambda_1 e^{\Sigma_1} \otimes \Sigma_2 P'_- e^{-\Sigma_1} \\ &\quad - i\alpha\beta\Sigma_2 \otimes P'_+ e^{-\Sigma_1} - i\alpha\beta\Lambda_2 e^{\Sigma_2} \otimes \Sigma_1 P'_- e^{-\Sigma_2} + i\alpha^2\beta\Sigma_1 H_1 \otimes P_+ e^{-(\Sigma_1+\Sigma_2)} \\ &\quad + i\alpha^2\beta H_2 \Lambda_1 e^{\Sigma_1} \otimes \Sigma_2 P_+ e^{-(\Sigma_1+\Sigma_2)} - i\alpha^2\beta P_+ e^{-(\Sigma_1+\Sigma_2)} \otimes \Sigma_1\Sigma_2 e^{-\frac{1}{2}(\Sigma_1+\Sigma_2)} \\ &\quad - i\alpha^2\beta\Sigma_2 H_2 \otimes P_+ e^{-(\Sigma_1+\Sigma_2)} - i\alpha^2\beta H_1\Lambda_2 e^{\Sigma_2} \otimes \Sigma_1 P_+ e^{-(\Sigma_1+\Sigma_2)} \\ &\quad + \alpha^2\beta^2\Sigma_1\Sigma_2 \otimes P_+ e^{-(\Sigma_1+\Sigma_2)} + \alpha^2\beta^2\Lambda_1\Lambda_2 e^{(\Sigma_1+\Sigma_2)} \otimes \Sigma_1\Sigma_2 P_+ e^{-(\Sigma_1+\Sigma_2)}\end{aligned} \tag{3.23}$$

Let us observe that in the purely Jordanian case ($\beta = 0$) all formulae (14)-(16) and (20)-(23) simplify significantly.

4 Outlook

The description in detail of all possible Hopf algebra structures of $D = 4$ quantum Lorentz algebras should have a physical importance. In particular the nonstandard deformations of the Lorentz algebra described by classical $r-$matrices satisfying CYBE are described by twist quantization and can be extended to the Poincaré algebra. Recently it has been shown that that the twist quantizations of Poincaré algebra is very useful in the construction of new noncommutative deformed Minkowski spaces which are covariant under relativistic symmetries see (e.g. [9]-[15]).

One can further observe that $o(3,1)$ describes as well the $D = 3$ de-Sitter algebra, which provides via quantum AdS contraction the $D = 3$ deformed Poincaré algebra. These novel aspects of the presented deformation as well as the structure of dual deformed Lorentz and Poincaré groups will be considered in our next publication.

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